COMP0114 Inverse Problems in Imaging. Coursework 1

Introduction

This coursework covers material in the first three weeks of the module. It is divided into three parts which roughly will correspond to the material in each week; however you can carry out the assignment in the time that suits you, with all parts being handed in as a single submission. The assignments will be closely related to the support in the lab classes which you should attend to get help with completing them.

Tasks

Week 1: 13. January

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1. Solving Underdetermined Problems (30%)

As explained in lectures the linear equation $x_1 + 2x_2 = 5$ is used to introduce the concept of minimum norm solutions of underdetermined problems:

$$\text{minimize} \quad \Phi = \sum_i |x_i|^p, \qquad \text{subject to} \quad Ax = b,$$

where in our case, $A = (1\ 2)$, $b = 5$.

a.) Write a function of two variables, x and p, where x is a vector of length 2 and p is a scalar; this function will compute the value of Φ as given above.

b.) Use library functions to compute solutions of the above optimization problem, for $p = 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5,\ 4$.

c.) Plot the solutions you have obtained as points on a 2D graph together with the line representing the constraint equation $x_1 + 2x_2 = 5$. The result should look like Figure 1.

d.) Another solution method is to use the *Moore-Penrose generalised inverse*

$$A^\dagger := A^\mathsf{T}(AA^\mathsf{T})^{-1}\,,$$

and then apply it to get $x_{\mathsf{MP}} = A^\dagger b$. Implement this solution and plot it on the same graph as in the previous question. What value of p does this correspond to and why?

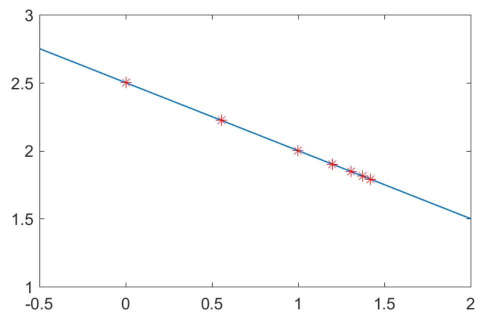

Figure 1: Example for a line plot for Exercise 1

In parts 2 and 3 of the assignment we will implement discrete convolution of a 1D function by an explict matrix vector multiplcation, analyse its behaviour and compare to other methods for performing convolution.

Week 2: 20 January

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2. Singular Value Decomposition (30%)

In this part we analyse the matrix representation of convolution using Singular Value Decomposition (SVD).

a.) Set up a spatial grid on the interval $[-1, 1]$ in n equally spaced steps of size δn. The grid represents the values $[x_1, \dots x_n]$ with $x_i = -1+(i-1)\delta n$ for $i = 1, \dots, n$ and $\delta n = 2/(n-1)$. Make sure that $x_1 = -1$ and $x_n = 1$.

b.) Create a vector of values of the Gaussian function centred at $\mu = 0$ with $\sigma = 0.2$, given by

$$G(x) = \frac{\delta n}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

You should evaluate this function at the grid points you create in part a).

c.) Create the convolution matrix of size $n \times n$ with entries

$$A_{i,j} = G(x_i - x_j) = \frac{\delta n}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i - x_j)^2}{2\sigma^2}\right) \tag{1}$$

d.) Plot the matrix A as an image, where A is considered as a 2D-array for the case $n = 100$.

e.) Compute the SVD of matrix A using library functions. You should obtain three matrices U, W, V, where W is a diagonal matrix of same size as A containing the singular values. Verify that the equation $A = UWV^{\mathsf{T}}$ is satisfied.

f.) Compute the pseudoinverse $A^\dagger$ of A by using the formula $A^\dagger = VW^\dagger U^\top$ as given in lectures. This requires you to create a method for constructing $W^\dagger$. For the case $n = 10$, check that this has the property $WW^\dagger = W^\dagger W = Id_n$ where Id_n is the $n \times n$ Identity matrix. Check also that $AA^\dagger = A^\dagger A = Id_n$.

g.) Repeat the last two steps for $n = 20$. What do you observe? Choose $n = 100$ again and plot the first 9 columns of V, the last 9 columns of V, and the singular values on a logarithmic scale, i.e. $\log(\mathrm{diag}(W))$.

Week 3: 27. January

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3. Convolutions and Fourier transform (40%)

In the following we want to examine the convolution of a 1D signal; we define a step function on the interval $[-1, 1]$ by

$$f(x) = \chi_{(-0.95,-0.6]}(x) + 0.2\chi_{(-0.6,-0.2]}(x) - 0.5\chi_{(-0.2,0.2]}(x) + 0.7\chi_{(0.4,0.6]}(x) - 0.7\chi_{(0.6,1]}(x),$$

where the characteristic function of an interval $(a, b]$ is defined as

$$\chi_{(a,b]}(x) = \begin{cases} 1 & \text{for} \quad a < x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Here's a plot of f in Figure 2.

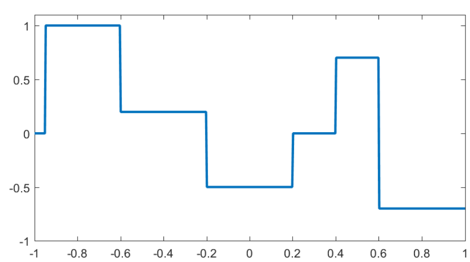

Figure 2: The step function f

a.) Create a function for f as above and plot it on a grid on the interval $[-1, 1]$. Choose a sufficiently large number of grid points n to resolve the jumps.

b.) Compute the matrix A as in part 2 for $\sigma = 0.05,\ 0.1,\ 0.2$ and plot the singular values.

c.) Verify that the plot of the singular values follows (half) a Gaussian function and determine the variance of this Gaussian in each case.

d.) Perform the convolution of the function f with the matrix A (by matrix multiplication) for all three choices of σ and plot the result.

e.) Since convolution is equivalent to multiplication in Fourier space, perform convolution by multiplication in Fourier space for the three choices of σ and plot the result (remember to take the inverse Fourier transform); comment on any differences that you observe.

f.) Repeat the convolution with the matrix A using periodic boundary conditions when assembling A.

Report

Write one report for all 3 parts. Explain your method and present your results and figures. Make sure that you provide an answer to all questions. The total length of the report would normally be between 6-10 pages. Submit your report using Moodle. Code can be uploaded separately.